Indefinite Backward Stochastic Linear-Quadratic Optimal Control Problems

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Outline

Problem formulation

State equation: Consider the BSDE over $[0, T]$:

$$\begin{cases} dY(t) = [A(t)Y(t) + B(t)u(t) + C(t)Z(t)]dt + Z(t)dW(t), \\ Y(T) = \xi, \end{cases}$$

- $A, C : [0, T] \to \mathbb{R}^{n\times n}$, and $B : [0, T] \to \mathbb{R}^{n\times m}$ are bounded and deterministic functions.
- W is a standard one-dimensional (for simplicity) BM.
- The control u belongs to the space

$$\mathscr{U} = \left\{ u : [0, T] \times \Omega \to \mathbb{R}^m \mid u \in \mathbb{F} \text{ and } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty \right\}.$$

- $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

Quadratic performance functional:

$$J(\xi; u) = \mathbb{E}\left[\langle GY(0), Y(0)\rangle + \int_0^T \left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle dt\right],$$

- The weighting matrices are bounded and deterministic, G and

$$\begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix}$$

  are symmetric, not required to be positive definite (semidefinite).

Problem (BSLQ). For a given terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, find a control $u^* \in \mathscr{U}$ such that

$$J(\xi; u^*) = \inf_{u \in \mathscr{U}} J(\xi; u) \equiv V(\xi).$$

Motivation

- The LQ optimal control problem for BSDEs was initially investigated by Lim–Zhou (2001, SICON) in the following form: To minimize

$$J(\xi; u) = \mathbb{E}\Big\{\langle GY(0), Y(0)\rangle + \int_0^T \Big[\langle QY, Y\rangle + \langle NZ, Z\rangle + \langle Ru, u\rangle\Big] dt\Big\},$$

subject to

$$\begin{cases} dY(t) = [A(t)Y(t) + B(t)u(t) + C(t)Z(t)]dt + Z(t)dW(t), \\ Y(T) = \xi, \end{cases}$$

where $H, Q, N \geqslant 0$, $R > 0$.

  - No cross terms in (Y, Z, u) appears in the cost functional.
  - $G, Q, N \geqslant 0$ and $R > 0$, standard definite problem.

- The general problem itself is interesting and challenging.
- Another motivation arises from differential game theory.

A zero-sum Stackelberg differential game

Consider the controlled linear SDE

$$\begin{cases} dX(t) = [AX + B_1u_1 + B_2u_2]dt + [CX + D_1u_1 + D_2u_2]dW, \\ X(0) = x, \end{cases}$$

and the performance functional (cost of Player 1, gain of Player 2)

$$J(x; u_1, u_2) = \mathbb{E}\Big\{ \big\langle GX(T), X(T) \big\rangle + 2\langle \xi, X(T) \rangle + \int_0^T \Big[\langle QX, X \rangle + \langle R_1u_1, u_1 \rangle + \langle R_2u_2, u_2 \rangle \Big] dt \Big\}.$$

Player 2 is the leader. She announces her control u_2. Player 1, the follower, solves an LQ optimal control problem. The Riccati equation for this LQ control problem is

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad - (PB_1 + C^\top PD_1)(R + D_1^\top PD_1)^{-1}(B_1^\top P + D_1^\top PC) = 0, \\ P(T) = G. \end{cases}$$

The minimum cost of Player 1 (w.r.t. u_2) is

$$V(u_2)=\mathbb{E}\Big\{\langle P(0)x,x\rangle+2\langle\eta(0),x\rangle+\int_0^T\Big[\langle(R_2+D_2^\top PD_2)u_2,u_2\rangle$$
$$+2\langle\eta,B_2u_2\rangle+2\langle\zeta,D_2u_2\rangle-\langle(R+D_1^\top PD_1)^{-1}v,v\rangle\Big]dt\Big\}.$$

where

$$\begin{aligned}v&=B_1^\top\eta+D_1^\top\zeta+D_1^\top PD_2u_2,\\ \Theta&=-(R+D_1^\top PD_1)^{-1}(B_1^\top P+D_1^\top PC),\end{aligned}$$

and (η,ζ) is the adapted solution of

$$\begin{cases}d\eta(t)=-\Big\{(A+B\Theta)^\top\eta+(C+D\Theta)^\top\zeta\\ \qquad\qquad+[(C+D\Theta)^\top PD_2+PB_2]u_2\Big\}dt+\zeta dW,\\ \eta(T)=\xi.\end{cases}$$

The leader's problem is then to choose u_2 in order to minimize

$$J(u_2) \triangleq -V(u_2).$$

Taking a deeper look, we see the problem of Player 2 is exactly the indefinite BSLQ problem we proposed, with cross terms in (Y, Z, u) in the cost functional.

An more specific example:

Consider

$$\max_{v\in L^2_{\mathbb{F}}(0,1;\mathbb{R})}\ \min_{u\in L^2_{\mathbb{F}}(0,1;\mathbb{R})} \mathbb{E}\left\{|X(1)|^2+2\xi X(1)+\int_0^1 \Big[|u(t)|^2-(a^2+1)|v(t)|^2\Big]dt\right\}$$

subject to

$$\begin{cases} dX(t) = u(t)dt + [X(t)+v(t)]dW(t), \quad t\in[0,1], \\ X(0) = 0, \end{cases}$$

where $L^2_{\mathbb{F}}(0,1;\mathbb{R})$ is the space of $\mathbb{F}$-progressively measurable processes $\varphi : [0,1]\times\Omega\to\mathbb{R}$ with $\mathbb{E}\int_0^1|\varphi(t)|^2dt<\infty$, ξ is an $\mathcal{F}_1$-measurable, bounded random variable, and $a>0$ is a constant.

For a given $v\in L^2_{\mathbb{F}}(0,1;\mathbb{R})$, the minimization problem is a standard forward stochastic LQ optimal control problem.

The minimum $V(\xi; v)$ (depending on ξ and v):

$$V(\xi; v) = \mathbb{E} \int_0^1 \Big[-|\eta(t)|^2 + 2\zeta(t)v(t) - a^2|v(t)|^2 \Big] dt,$$

where (η, ζ) is the adapted solution to the BSDE

$$\begin{cases} d\eta(t) = [\eta(t) - \zeta(t) - v(t)]dt + \zeta(t)dW(t), \quad t \in [0, 1], \\ \eta(1) = \xi. \end{cases}$$

Using the transformations

$$Y(t) = \eta(t), \quad Z(t) = \zeta(t), \quad u(t) = v(t) - \frac{1}{a^2}\zeta(t),$$

we see the maximization problem is equivalent to the BSLQ problem with the state equation

$$\begin{cases} dY(t) = \Big[Y(t) - \frac{a^2 + 1}{a^2} Z(t) - u(t) \Big] dt + Z(t)dW(t), \quad t \in [0, 1], \\ Y(1) = \xi \end{cases}$$

and the cost functional

$$J(\xi; v) = \mathbb{E} \int_0^1 \Big[|Y(t)|^2 - \frac{1}{a^2}|Z(t)|^2 + a^2|u(t)|^2 \Big] dt.$$

Outline

Existence of an optimal control

Theorem. For a given terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, a control $u^* \in \mathscr{U}$ is optimal iff the following conditions hold:

1. $J(0;u) \geqslant 0$ for all $u \in \mathscr{U}$.
2. The adapted solution (X^*, Y^*, Z^*) to the decoupled FBSDE

$$\begin{cases} dX^*(t) = (-A^\top X^* + QY^* + S_1^\top Z^* + S_2^\top u^*)dt \\ \qquad\qquad + (-C^\top X^* + S_1 Y^* + R_{11} Z^* + R_{12} u^*)dW, \\ dY^*(t) = (AY^* + Bu^* + CZ^*)dt + Z^* dW, \\ X^*(0) = GY^*(0), \quad Y^*(T) = \xi, \end{cases}$$

satisfies

$$S_2 Y^* + R_{21} Z^* - B^\top X^* + R_{22} u^* = 0.$$

Proof. $u^* \in \mathscr{U}$ is optimal for ξ iff

$$J(\xi; u^* + \varepsilon u) - J(\xi; u^*) \geqslant 0, \quad \forall u \in \mathscr{U}, \ \forall \varepsilon \in \mathbb{R}.$$

A straightforward computation yields

$$\begin{aligned} J(\xi; u^* + \varepsilon u) - J(\xi; u^*) &= \varepsilon^2 J(0; u) \\ &+ 2\varepsilon \mathbb{E}\Bigg[\langle GY^*(0), Y(0)\rangle + \int_0^T \Bigg\langle \begin{pmatrix} Q & S_1^\top & S_2^\top \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} Y^* \\ Z^* \\ u^* \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ u \end{pmatrix} \Bigg\rangle dt\Bigg]. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} -\langle GY^*(0), Y(0)\rangle &= -\langle X^*(0), Y(0)\rangle \\ &= \mathbb{E}\int_0^T \Big[\langle QY^* + S_1^\top Z^* + S_2^\top u^*, Y\rangle \\ &\quad + \langle S_1 Y^* + R_{11} Z^* + R_{12} u^*, Z\rangle + \langle B^\top X^*, u\rangle\Big] dt. \end{aligned}$$

Upon substitution, we get

$$J(\xi; u^* + \varepsilon u) - J(\xi; u^*) = \varepsilon^2 J(0; u) + 2\varepsilon \mathbb{E}\int_0^T \langle S_2 Y^* + R_{21} Z^* - B^\top X^* + R_{22} u^*, u\rangle dt.$$

Outline

Construction of optimal controls

The natural idea: Solve for u^* from the FBSDE

$$\begin{cases} dX^*(t) = (-A^\top X^* + QY^* + S_1^\top Z^* + S_2^\top u^*)dt \\ \qquad\qquad + (-C^\top X^* + S_1 Y^* + R_{11} Z^* + R_{12} u^*)dW, \\ dY^*(t) = (AY^* + Bu^* + CZ^*)dt + Z^* dW, \\ X^*(0) = GY^*(0), \quad Y^*(T) = \xi, \end{cases}$$

coupled by

$$S_2 Y^* + R_{21} Z^* - B^\top X^* + R_{22} u^* = 0.$$

The basic method: Decoupling by the ansatz

$$Y^*(t) = -\Sigma(t) X^*(t) + \varphi(t),$$

where Σ is a deterministic function with $\Sigma(T) = 0$, and φ is stochastic process with $\varphi(T) = \xi$.

How to decide Σ and φ?

How to decide Σ and φ? Differentiating both sides of

$$Y^*(t) = -\Sigma(t)X^*(t) + \varphi(t),$$

comparing the coefficients of the drift and the diffusion, and using the relation

$$S_2 Y^* + R_{21} Z^* - B^\top X^* + R_{22} u^* = 0$$

to eliminate u^* (under certain assumptions on Σ), we will see that

- Σ satisfies a complicated ODE;
- φ satisfies a BSDE whose coefficients depend on Σ:

$$\begin{cases} d\varphi(t) = \alpha(t; \Sigma)dt + \beta(t)dW(t), \quad t \in [0, T], \\ \varphi(T) = \xi; \end{cases}$$

- Z is linear combination of X, φ, and β;
- u^* is a linear combination of X and φ.

The fundamental question: Does such a Σ exists? In other words, is the deduced ODE solvable?

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Unfortunately, in the general case $J(0; u) \geqslant 0$, even an optimal control exists, the decoupling method might not work!

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Unfortunately, in the general case $J(0; u) \geqslant 0$, even an optimal control exists, the decoupling method might not work!

The uniform convexity condition:

$$J(0; u) \geqslant \delta \mathbb{E} \int_0^T |u(t)|^2 dt, \quad \forall u \in \mathscr{U},$$

stronger than $J(0; u) \geqslant 0$, but not too much. It can be easily shown that under the uniform convexity condition, an optimal control uniquely exists.

- Does the decoupling method work in the uniform convexity case? It has been shown by Lim–Zhou, that in the definite case, a special uniform convexity condition, the decoupling method works. The argument is highly dependent on the two assumptions:
  - No cross terms in (Y, Z, u) appears in the cost functional.
  - $H, Q, N \geqslant 0$ and $R > 0$, standard definite problem.
- If the decoupling method works in the uniform convexity case, how can we use the result to solve the general case $J(0; u) \geqslant 0$?

Consider, for each $\varepsilon > 0$, the new cost functional $J_\varepsilon(\xi; u)$ defined by

$$J_\varepsilon(\xi; u) = J(\xi; u) + \varepsilon\, \mathbb{E} \int_0^T |u(t)|^2 dt,$$

which is uniform convex when $J(0; u) \geqslant 0$. Suppose that we can construct the (unique) optimal control u_ε^* for $J_\varepsilon(\xi; u)$.

Theorem. For the original problem, an optimal control exists for a given terminal state ξ iff one of the following conditions holds:

(i) the family $\{u_\varepsilon^*\}_{\varepsilon>0}$ is bounded in the Hilbert space $\mathscr{U}$, i.e.,

$$\sup_{\varepsilon>0} \mathbb{E} \int_0^T |u_\varepsilon^*(t)|^2 dt < \infty.$$

(ii) u_ε^* converges weakly in $\mathscr{U}$ as $\varepsilon \to 0$;

(iii) u_ε^* converges strongly in $\mathscr{U}$ as $\varepsilon \to 0$.

Whenever (i), (ii), or (iii) is satisfied, the strong (weak) limit $u^* = \lim_{\varepsilon\to 0} u_\varepsilon^*$ is an optimal control for ξ.

Outline

Connections with FSLQ problems

Consider the controlled linear forward SDE

$$\begin{cases} dX(t) = [A(t)X(t) + B(t)u(t) + C(t)v(t)]dt + v(t)dW(t), \quad t \in [0, T], \\ X(0) = x, \end{cases}$$

and, for $\lambda > 0$, the cost functional

$$\mathcal{J}_\lambda(x; u, v) \triangleq \mathbb{E}\Big\{ \lambda |X(T)|^2 + \int_0^T \Big\langle \begin{pmatrix} Q & S_1^\top & S_2^\top \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} X \\ v \\ u \end{pmatrix}, \begin{pmatrix} X \\ v \\ u \end{pmatrix} \Big\rangle dt \Big\}.$$

In the above, the control is the pair

$$(u, v) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times L^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \equiv \mathscr{U} \times \mathscr{V}.$$

Problem (FSLQ)$_\lambda$. For a given initial state $x \in \mathbb{R}^n$, find a control $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ such that

$$\mathcal{J}_\lambda(x; u^*, v^*) = \inf_{(u,v) \in \mathscr{U} \times \mathscr{V}} \mathcal{J}_\lambda(x; u, v) \equiv \mathcal{V}_\lambda(x).$$

Recall the **uniform convexity condition**:

$$J(0;u) \geqslant \delta\,\mathbb{E}\int_0^T |u(t)|^2 dt, \quad \forall u \in \mathscr{U},$$

Theorem. Assume that the uniform convexity condition holds. Then there exist constants $\rho > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$,

$$\mathcal{J}_\lambda(0;u,v) \geqslant \rho\,\mathbb{E}\int_0^T \Big[|u(t)|^2 + |v(t)|^2\Big] dt, \quad \forall (u,v) \in \mathscr{U} \times \mathscr{V}.$$

If, in addition, $G = 0$, then for $\lambda \geqslant \lambda_0$,

$$\mathcal{J}_\lambda(x;u,v) \geqslant \rho\,\mathbb{E}\int_0^T \Big[|u(t)|^2 + |v(t)|^2\Big] dt, \quad \forall (u,v) \in \mathscr{U} \times \mathscr{V},\ \forall x \in \mathbb{R}^n.$$

Corollary. Under the assumptions of the previous theorem, for $\lambda \geqslant \lambda_0$,

1. Problem $(\text{FSLQ})_\lambda$ is uniquely solvable. If, in addition, $G = 0$, then the value function $\mathcal{V}_\lambda$ satisfies

$$\mathcal{V}_\lambda(x) \geqslant 0, \quad \forall x \in \mathbb{R}^n.$$

2. the Riccati equation

$$\begin{cases} \dot{P}_\lambda + P_\lambda A + A^\top P_\lambda + Q \\ \qquad - \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix}^\top \begin{pmatrix} R_{11} + P_\lambda & R_{12} \\ R_{21} & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix} = 0, \\ P_\lambda(T) = \lambda I, \end{cases}$$

admits a unique solution $P_\lambda \in C([0,T]; \mathbb{S}^n)$ such that

$$\begin{pmatrix} R_{11} + P_\lambda & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \gg 0.$$

Moreover, $\mathcal{V}_\lambda(x) = \langle P_\lambda(0)x, x\rangle$ for all $x \in \mathbb{R}^n$.

Remark. (ii) implies $R_{22} \gg 0$.

Outline

Properties of P_λ

The Riccati equation

$$\begin{cases} \dot{P}_\lambda + P_\lambda A + A^\top P_\lambda + Q \\ \qquad - \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix}^\top \begin{pmatrix} R_{11} + P_\lambda & R_{12} \\ R_{21} & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix} = 0, \\ P_\lambda(T) = \lambda I, \end{cases}$$

Recall the decoupling relation

$$Y^*(t) = -\Sigma(t) X^*(t) + \varphi(t).$$

Hope to show that

$$\Sigma(t) = \lim_{\lambda \to \infty} P_\lambda(t)^{-1}.$$

- Is $P_\lambda(t)$ invertible?
- Does $P_\lambda(t)^{-1}$ converge?

Let us temporarily assume that

$$G = 0, \quad Q(t) = 0, \quad R_{12}(t) = R_{21}^{\top}(t) = 0; \quad \forall t \in [0, T], \tag{1}$$

i.e., the cost functional takes the form

$$\begin{aligned} J(\xi; u) &= \mathbb{E}\int_0^T \left\langle \begin{pmatrix} 0 & S_1^{\top}(t) & S_2^{\top}(t) \\ S_1(t) & R_{11}(t) & 0 \\ S_2(t) & 0 & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle dt \\ &= \mathbb{E}\int_0^T \Big[2\langle S_1 Y, Z\rangle + 2\langle S_2 Y, u\rangle + \langle R_{11} Z, Z\rangle + \langle R_{22} u, u\rangle \Big] dt. \end{aligned}$$

Proposition. Let (1) hold. Then for $\lambda \geqslant \lambda_0$,

$$P_\lambda(t) \geqslant 0, \quad \forall t \in [0, T].$$

Moreover, for every $\lambda_2 > \lambda_1 \geqslant \lambda_0$, we have

$$P_{\lambda_2}(t) > P_{\lambda_1}(t), \quad \forall t \in [0, T].$$

Write for an $\mathbb{S}^n$-valued function $\Sigma : [0, T] \to \mathbb{S}^n$,

$$\mathcal{B}(t, \Sigma(t)) = B(t) + \Sigma(t)S_2(t)^\top,$$
$$\mathcal{C}(t, \Sigma(t)) = C(t) + \Sigma(t)S_1(t)^\top,$$
$$\mathcal{R}(t, \Sigma(t)) = I + \Sigma(t)R_{11}(t).$$

The Riccati equation for Problem (BSLQ):

$$\begin{cases} \dot{\Sigma}(t) - A(t)\Sigma(t) - \Sigma(t)A(t)^\top + \mathcal{B}(t, \Sigma(t))[R_{22}(t)]^{-1}\mathcal{B}(t, \Sigma(t))^\top \\ \qquad + \mathcal{C}(t, \Sigma(t))[\mathcal{R}(t, \Sigma(t))]^{-1}\Sigma(t)\mathcal{C}(t, \Sigma(t))^\top = 0, \\ \Sigma(T) = 0. \end{cases}$$

Theorem. Let (1) hold. Then the above Riccati equation admits a unique positive semidefinite solution $\Sigma \in C([0, T]; \mathbb{S}^n)$ such that $\mathcal{R}(\Sigma)$ is invertible a.e. on $[0, T]$ and $\mathcal{R}(\Sigma)^{-1} \in L^\infty(0, T; \mathbb{R}^n)$.

Theorem. Let (1) hold. Let (φ, β) be the adapted solution to the BSDE

$$\begin{cases} d\varphi(t) = \Big\{[A - \mathcal{B}(\Sigma)R_{22}^{-1}S_2 - \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1]\varphi \\ \qquad\qquad + \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta\Big\}dt + \beta dW(t), \\ \varphi(T) = \xi. \end{cases}$$

and X the solution to the following SDE:

$$\begin{cases} dX(t) = \Big\{\big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top - A^\top\big]X \\ \qquad\qquad - \big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi + S_1^\top \mathcal{R}(\Sigma)^{-1}\beta\Big\}dt \\ \qquad\qquad - \big[\mathcal{R}(\Sigma)^{-1}\big]^\top\big[\mathcal{C}(\Sigma)^\top X - S_1\varphi - R_{11}\beta\big]dW(t), \\ X(0) = 0. \end{cases}$$

Then the optimal control of Problem (BSLQ) for the terminal state ξ is given by

$$u(t) = [R_{22}(t)]^{-1}[\mathcal{B}(t, \Sigma(t))^\top X(t) - S_2(t)\varphi(t)], \quad t \in [0, T],$$

and the value function of Problem (BSLQ) is given by

$$V(\xi) = \mathbb{E}\int_0^T \Big\{\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta, \beta\rangle + 2\langle S_1^\top \mathcal{R}(\Sigma)^{-1}\beta, \varphi\rangle \\ - \langle [S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2]\varphi, \varphi\rangle\Big\}dt.$$

Outline

Transformation

State equation:

$$\begin{cases} dY(t) = (AY + Bu + CZ)dt + ZdW(t), \\ Y(T) = \xi, \end{cases}$$

Quadratic performance functional:

$$J(\xi; u) = \mathbb{E}\left[\langle GY(0), Y(0)\rangle + \int_0^T \left\langle \begin{pmatrix} Q & S_1^\top & S_2^\top \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} Y \\ Z \\ u \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ u \end{pmatrix} \right\rangle dt\right],$$

Recall that when the uniform-convexity condition holds, $R_{22} \gg 0$. So we can eliminate the crossing term in u and Z by proper transformations.

Let

$$\mathscr{S}_1 = S_1 - R_{12}R_{22}^{-1}S_2, \qquad \mathscr{R}_{11} = R_{11} - R_{12}R_{22}^{-1}R_{21},$$
$$\mathscr{C} = C - BR_{22}^{-1}R_{21}, \qquad v = u + R_{22}^{-1}R_{21}Z,$$

The original Problem (BSLQ) then is equivalent to the BSLQ problem with state equation

$$\begin{cases} dY(t) = (AY + Bv + \mathscr{C}Z)dt + ZdW(t), \\ Y(T) = \xi, \end{cases}$$

and cost functional

$$\mathscr{J}(\xi; v) = \mathbb{E}\Big\{\langle GY(0), Y(0)\rangle + \int_0^T \Big\langle \begin{pmatrix} Q & \mathscr{S}_1^\top & S_2^\top \\ \mathscr{S}_1 & \mathscr{R}_{11} & 0 \\ S_2 & 0 & R_{22} \end{pmatrix} \begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \Big\rangle dt \Big\}.$$

Furthermore, let $H \in C([0,T];\mathbb{S}^n)$ be the solution to the ODE

$$\begin{cases} \dot{H}(t) + H(t)A(t) + A(t)^\top H(t) + Q(t) = 0, \quad t \in [0,T], \\ H(0) = G, \end{cases}$$

Apply the integration by parts formula to $t \mapsto \langle H(t)Y(t), Y(t)\rangle$:

$$\begin{aligned} &\mathbb{E}\langle H(T)\xi,\xi\rangle - \mathbb{E}\langle GY(0),Y(0)\rangle \\ &= \mathbb{E}\int_0^T \left\langle \begin{pmatrix} -Q & H\mathscr{C} & HB \\ \mathscr{C}^\top H & H & 0 \\ B^\top H & 0 & 0 \end{pmatrix} \begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \right\rangle dt. \end{aligned}$$

Substituting for $\mathbb{E}\langle GY(0),Y(0)\rangle$ in $\mathscr{J}(\xi;v)$ yields

$$\mathscr{J}(\xi;v) = \mathbb{E}\int_0^T \left\langle \begin{pmatrix} 0 & (S_1^H)^\top & (S_2^H)^\top \\ S_1^H & R_{11}^H & 0 \\ S_2^H & 0 & R_{22} \end{pmatrix} \begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \right\rangle dt - \mathbb{E}\langle H(T)\xi,\xi\rangle,$$

where

$$S_1^H = \mathscr{S}_1 + \mathscr{C}^\top H, \quad S_2^H = S_2 + B^\top H, \quad R_{11}^H = \mathscr{R}_{11} + H.$$

Thus, for a given terminal state ξ,the original problem is equivalent to minimizing the cost functional

$$J^H(\xi; v) = \mathbb{E}\int_0^T \Big\langle \begin{pmatrix} 0 & (S_1^H)^\top & (S_2^H)^\top \\ S_1^H & R_{11}^H & 0 \\ S_2^H & 0 & R_{22} \end{pmatrix}\begin{pmatrix} Y \\ Z \\ v \end{pmatrix}, \begin{pmatrix} Y \\ Z \\ v \end{pmatrix} \Big\rangle dt,$$

subject to the state equation

$$\begin{cases} dY(t) = (AY + Bv + \mathscr{C}Z)dt + ZdW(t), \\ Y(T) = \xi. \end{cases}$$

Remark. For BSLQ problems, the presence of crossing terms in (Y, Z), (Y, u) is essential.

Thanks For Your Attention